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A QUALITATIVE ANALYSIS OF THE MOTION OF A HEAVY SOLID OF REVOLUTION ON AN ABSOLUTELY ROUGH PLANE*

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A strictly convex heavy solid of revolution, moving without slipping on a horizontal plane in a uniform gravitational field, is considered. Thanks to the existence of three first integrals (the explicit form of two of which is not known), the motion is amenable to qualitative analysis. The variation of the angle of nutation is studied and the motion of the point of contact on the surface of the body and on the supporting plane is analysed. It is shown that in phase space there are three-dimensional tori with conditionally-periodic motions. The problem of the motion of a body similar in shape and mass distribution to a dynamically and geometrically symmetric body is considered. Generalizations of KAM-theory to reversible systems are used to establish the conservation of the majority of invariant tori.

The problem of the rolling of a heavy body of revolution was first studied by Chaplygin /1/. The investigation was carried further in /2/. Up to now, fairly detailed attention has been devoted to the existence and stability of steady motions of a solid of revolution on an absolutely rough plane /3-8/. A qualitative analysis has been carried out /9/ of the motion without slipping of a heavy homogeneous tri-axial ellipsoid on a horizontal plane, on the assumption that it is similar to a sphere, and the periodic motions of the ellipsoid have been studied /10/.

1. Let $O\xi\eta\zeta$ be a fixed coordinate system with origin at the point O on a horizontal plane $O\xi\eta$, on which a body is moving (the $O\zeta$ axis points vertically upward), and let $Gxyz$ be a coordinate system fixed in the body. The origin of the body coordinate system is at its centre of gravity, and the axes lie along its principal central axes of inertia. The mutual orientation of the body and fixed coordinate systems is defined in terms of the Euler angles ψ, θ, φ. We also introduce a moving coordinate system $Q\xi_1\eta_1\zeta_1$, whose axes are parallel to the ξ, η, ζ axes, respectively, and whose origin Q is the projection of the centre of gravity G onto the horizontal plane.

As coordinates defining the position of the body we take the Euler angles and the two coordinates ξ, η of the centre of gravity G in the coordinate system $O\xi\eta\zeta$. The third coordinate ζ - the height of the centre of gravity above the supporting plane - is a function of the angles θ, φ, given the shape of the surface bounding the body.

To non-integrable constraints are imposed on the system: the horizontal component of the absolute velocity of the point P of the body coinciding with the point of contact is zero, and the entire mechanical system is a conservative non-holonomic system in Chaplygin's sense /1/, with three degrees of freedom. Its dynamic equations (describing the rotation of the rigid body about its centre of gravity G) are separated and can be considered independently of the constraint equations.

In canonical form these equations are

$$\dot{\mathbf{q}} = \partial H/\partial \mathbf{p}, \quad \mathbf{p}^{\cdot} = -\partial H/\partial \mathbf{q} + \mathbf{\Gamma}, \quad \mathbf{q}^T = (\theta, \psi, \varphi), \quad \mathbf{p}^T = (p_\theta, p_\psi, p_\varphi) \tag{1.1}$$

The function H in (1.1) is the result of applying the Legendre transformation to the Lagrangian L, where the latter is assumed to incorporate the constraints on the body, $\mathbf{p} = \partial L/\partial \mathbf{q}^{\cdot}$ and $\mathbf{\Gamma}$ are the non-holonomicity terms.

Eqs.(1.1) define a reversible flow on T^*M, where $M = SO(3)$ is the set of positions of the reduced system. They are fairly complicated to investigate and may involve effects not occurring in Hamiltonian systems. For example, in the general case, Eqs.(1.1) do not have an invariant measure /11/. In addition, they admit of steady solutions, which may be asymptotically stable with respect to some of the variables /8/.

We shall use the following notation: m is the mass of the body, g the acceleration due to gravity, and A, B and C the moments of inrertia of the body relative to the x, y and z axes respectively.

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2. Let us assume that the surface bounding the body is a surface of revolution about the z axis, and that the body is dynamically symmetric, i.e., $A = B$, $\zeta = f(\theta)$. We shall assume that f is a thrice continuously differentiable function of θ. Note that $f(\theta)$ may be assumed to be an even, 2π-periodic function of its argument, and moreover $r(\theta) \equiv f(\theta) + f''(\theta) > 0$. Here and below the prime will denote differentiation with respect to θ. Clearly, $f'(0) = f'(\pi) = 0$.

The function H will have the form

$$H = \frac{1}{2}\langle \mathbf{p}, \boldsymbol{\omega}\mathbf{p}\rangle + mgf(\theta); \quad \boldsymbol{\omega}(\theta) = \begin{Vmatrix} \omega_1 & 0 \\ 0 & \Delta^{-1}\boldsymbol{\omega}_2 \end{Vmatrix} \tag{2.1}$$

$$\boldsymbol{\omega}_2(\theta) = \begin{Vmatrix} C + m\mu^2 & -C\cos\theta - mf'\mu \\ -C\cos\theta - mf'\mu & A\sin^2\theta + C\cos^2\theta + mf'^2 \end{Vmatrix}$$

$$\omega_1(\theta) = [A + m(f^2 + f'^2)]^{-1}, \quad \Delta(\theta) = AC\delta^2\sin^2\theta,$$

$$\delta(\theta) = \left(1 + \frac{m\mu^2}{C} + \frac{m\varkappa^2}{A}\right)^{1/2} > 0$$

$$\mu(\theta) = f\sin\theta + f'\cos\theta, \quad \varkappa(\theta) = -f\cos\theta + f'\sin\theta$$

Here $\boldsymbol{\omega}$ is a (3×3) symmetric matrix, the angular brackets denote the scalar product, μ and $\varkappa$ are the coordinates of the point of contact P in the coordinate system Gx_1z, where the x_1 axis lies in the plane of the vertical meridian of the body, perpendicular to the z axis.

The differential Eqs.(1.1) will have an invariant measure, defined by the density $\delta^{-1}(\theta)$.

We now write down the equations for p_ψ and p_φ, using the abbreviated notation $\mathbf{p}_*^T = (p_\psi, p_\varphi)$:

$$p_*^{\cdot} = \theta^{\cdot}\mathrm{S}(\theta)\sin\theta\mathbf{p}_*; \ \mathrm{S} = \| s_{jk} \| \ (j, k = 1, 2, 3) \tag{2.2}$$

$$s_{11} = e_1(Ce_2 + m\mu^2 ff'), \quad s_{12} = e_1(A\mu f''\sin^2\theta - Ce_2\cos\theta - m\mu ff'^2)$$

$$s_{21} = e_1(Ce_3 + m\mu^3 f), \quad s_{22} = e_1(A\mu r\sin^2\theta\cos\theta - Ce_3\cos\theta - m\mu^2 ff')$$

$$e_1(\theta) = m\Delta^{-1}\sin^{-1}(\theta), \quad e_2(\theta) = ff' + f''\varkappa\sin\theta, \quad e_3(\theta) = \mu f + r\varkappa\sin\theta\cos\theta$$

The elements of the matrix S are 2π-periodic, continuously differentiable even functions of θ.

We now transform (2.2) to the new independent variable θ. This gives the following linear differential equations with periodic coefficients:

$$\mathbf{p}_*' = \mathrm{S}(\theta)\sin\theta\mathbf{p}_* \tag{2.3}$$

which have the θ-invariance property /12/. Let $X(\theta)$ be the matriciant of system (2.3). It exists, and its elements are continuously differentiable functions of θ. Note that $\det\mathbf{X} = \delta(\theta)/\delta(0) > 0$ (the Liouville-Jacobi formula). System (2.3) is invariant under rotations of θ, and therefore the spectrum of the monodromy matrix $\mathbf{X}(2\pi)$ is symmetric about the unit circle and the real axis.

By the Floquet-Lyapunov theory, the matriciant of system (2.3) can be expressed as

$$\mathbf{X}(\theta) = \mathbf{F}(\theta)\exp(\theta\mathbf{K}) \tag{2.4}$$

where $\mathbf{F}(\theta)$ is a 2π-periodic (or antiperiodic) real continuously differentiable non-singular matrix and $\mathbf{K}$ is a real constant matrix. Each element of the matrix $\exp(\theta\mathbf{K})$ is a linear combination either of $\cos\alpha\theta, \sin\alpha\theta$, or of $e^{\alpha\theta}, e^{-\alpha\theta}$, or at $\theta, 1$.

Note that if the body is bounded by a spherical surface, i.e., $f(\theta) = r + d\cos\theta$ ($|d| < r$; r, $d = \mathrm{const}$), then $X(\theta)$ is a known 2π-periodic matrix /1/. Hence we have

Proposition 1. System (1.1) in this case admits of two independent integrals P_1, P_2, which are linear functions of the momenta p_ψ and p_φ:

$$\mathbf{P}_* = \mathbf{X}^{-1}(\theta)\mathbf{p}_*, \quad \mathbf{P}_*^T = (P_1, P_2) \tag{2.5}$$

Thanks to the existence of three first integrals H, P_1, P_2, Eqs.(1.1) enable us to carry out a qualitative analysis of the motion.

We transform Eqs.(1.1) from p_ψ and p_φ to the new "momenta" P_1 and P_2. In terms of the new variables, the function H will be

$$H_0 = {}^1/_2\omega_1 p_\theta^2 + \Pi(\theta, P_1, P_2) \tag{2.6}$$

$$\Pi(\theta, P_1, P_2) = {}^1/_2\Delta^{-1}(d_{11}P_1^2 + d_{12}P_1P_2 + d_{22}P_2^2) + mgf$$

$$d_{ij}(\theta) = \langle x_i, \boldsymbol{\omega}_2 x_j\rangle, \quad \mathbf{x}_1^T = (x_{11}, x_{21}), \quad \mathbf{x}_2^T = (x_{12}, x_{22})$$

Note that the equations for ϑ and p_θ now become

$$\theta^{\cdot} = \partial H_0/\partial p_\theta,\ p_\theta^{\cdot} = -\partial H_0/\partial\theta$$

The generalized velocities $\psi^{\cdot}$ and $\varphi^{\cdot}$ and the new momenta P_1, P_2 are related as follows:

$$\psi^{\cdot} = \Delta^{-1}(\rho_1 P_1 + \rho_2 P_2),\ \varphi^{\cdot} = \Delta^{-1}(\rho_3 P_1 + \rho_4 P_2) \tag{2.7}$$

$$\rho(\theta) = \begin{Vmatrix} \rho_1 & \rho_2 \\ \rho_3 & \rho_4 \end{Vmatrix}, \quad \rho = \omega_2 \mathbf{X}$$

Investigation of the motion now reduces to a consideration of a reduced Hamiltonian system with one degree of freedom with Hamiltonian H_0, potential energy Π and kinetic energy $T = 1/2\omega_1^{-1}\theta^{\cdot 2}$. Further analysis of the motion has much in common with the analysis in /13/ of the motion of a dynamically and geometrically symmetric body on an absolutely smooth plane. We shall therefore confine ourselves to a brief statement of the main results.

The variation of the angle $\theta = \theta(t)$ is found with the help of the energy integral $T + \Pi = h = \text{const}$. Letting θ_0 denote the initial value of θ and disregarding motions for which $\theta = \theta_0$, we obtain

$$\pm\int_{\theta_0}^{\theta} [2\omega_1(h - \Pi)]^{-1/2}\, d\theta = t$$

In the set of levels of the first integrals $H_0 = h,\ P_1 = c_1,\ P_2 = c_2$, we define three subsets:

$$\Sigma = \{h = \Pi(\theta, c_1, c_2),\ \Pi'(\theta, c_1, c_2) = 0\},\ \Sigma_1 = \{c_1 - c_2 = 0\}$$
$$\Sigma_2 = \{c_1[x_{11}(\pi) + x_{21}(\pi)] + c_2[x_{12}(\pi) + x_{22}(\pi)] = 0\}$$

Σ is the set of critical values of the integral transformation (bifurcation set), the condition $\mathbf{c} \equiv (h, c_1, c_2) \in \Sigma_1$ ($\mathbf{c} \in \Sigma_2$) is a necessary condition for the angle θ to equal 0 (π) during the motion.

It is clear that $h - \Pi \geqslant 0$. This inequality determines the region of possible motions of the reduced one-dimensional system.

1°. Let $\mathbf{c} \notin \Sigma_1 \cup \Sigma_2$. Then the value of $h - \Pi(\theta, c_1, c_2)$ becomes negative if $\theta \to 0$ or π. Consequently, the angle θ lies between the two real roots of the equation $h - \Pi = 0$ in the interval $(0, \pi)$. If $\mathbf{c} \notin \Sigma$, then all roots of this equation are simple. Let $\theta_1(\mathbf{c})$ and $\theta_2(\mathbf{c})$ be two different roots of the equation $h - \Pi = 0$ and assume that $h > \Pi$ in the interval between them. Then the angle θ oscillates between θ_1 and θ_2 and the period of the oscillations is

$$\tau = 2\int_{\theta_1}^{\theta_2} [2\omega_1(h - \Pi)]^{-1/2}\, d\theta \tag{2.8}$$

2°. Let $\mathbf{c} \in \Sigma_1, \mathbf{c} \notin \Sigma_2, \mathbf{c} \notin \Sigma$. Then the axis of symmetry may pass through the vertical position. The angle θ will again oscillate between certain limits θ_1 and θ_2 with $\pi > \theta_2 > \theta_1 > -\pi$.

3°. Let $\mathbf{c} \in \Sigma_2, \mathbf{c} \notin \Sigma_1, \mathbf{c} \notin \Sigma$. The axis of symmetry may pass through the position $\theta = \pi$ (flipping). The angle θ will oscillate between θ_1 and θ_2, with $2\pi > \theta_2 > \theta_1 > 0$.

4°. If $\mathbf{c} \in \Sigma_1 \cap \Sigma_2, \mathbf{c} \notin \Sigma$, i.e., $c_1 = c_2 = 0$, then as the body moves its axis of symmetry may pass through both singular positions $\theta = 0$ and $\theta = \pi$. It follows from (2.7) that in this case $\psi = \psi_0 = \text{const}, \varphi = \varphi_0 = \text{const}$, i.e., the body moves in such a way that its axis of symmetry remains constantly in a fixed vertical plane of the moving coordinate system $Q\xi_1\eta_1\zeta_1$. The possibilities for the motion are either oscillations (relative to the angle θ) or rotations. The constraint equations

$$\xi^{\cdot} = -(f'\psi^{\cdot} + \mu\varphi^{\cdot})\cos\psi + f\theta^{\cdot}\sin\psi$$
$$\eta^{\cdot} = -(f'\psi^{\cdot} + \mu\varphi^{\cdot})\sin\psi - f\theta^{\cdot}\cos\psi$$

imply that at this level of the first integrals

$$\xi^{\cdot} = f\theta^{\cdot}\sin\psi_0,\ \eta^{\cdot} = -f\theta^{\cdot}\cos\psi_0 \tag{2.9}$$

Hence

$$\xi\cos\psi_0 + \eta\sin\psi_0 = \text{const} \tag{2.10}$$

$$\xi = \sin\psi_0\int_{\theta_0}^{\theta} f\, d\theta + \xi_0, \quad \eta = -\cos\psi_0\int_{\theta_0}^{\theta} f\, d\theta + \eta_0 \tag{2.11}$$

Consequently, the point Q either oscillates about the straight line (2.10) or goes to infinity along the line. Analogous arguments hold for the point P, except that f must be replaced by r in (2.9), (2.11).

5°. If $c \in \Sigma$, this is a singular level of the first integrals and the possibilities are either motions with constant angle θ (in these motions the integrals H, P_1, P_2 are dependent) or motions asymptotic to the latter. Motions with $\theta = \theta_0 = \mathrm{const}$ are either regular precession, rolling along a straight line or equilibrium positions.

If $\theta = \theta(t)$ is known, the variation of the angles $\psi = \psi(t), \varphi = \varphi(t)$ is found from (2.7) by quadratures. At any non-singular level of the first integrals, $\theta(t)$ is a periodic function of time (of period τ). Let

$$\lambda_1 = 2\pi/\tau \tag{2.12}$$

In time τ the angles ψ and φ receive certain constant increments λ_2 and λ_3, and we have

$$\psi = \lambda_2 t + \psi_*(t),\ \varphi = \lambda_3 t + \varphi_*(t)$$

The constants $\lambda_1, \lambda_2, \lambda_3$ (the frequencies of motion) depend on the constants of the first integrals, while the functions $\psi_*(t), \varphi_*(t)$ are periodic, with the same period τ.

Investigation of the curves traced out by the point of contact on the surface of the body and the moving plane $Q\xi_1\eta_1$ is entirely analogous /13/. A significant departure from the case considered in /13/ appears when one studies the curve traced out by the point of contact on the supporting - fixed - plane.

We shall investigate the motion of the point of contact on the supporting plane, following /14/. Let ξ_P and η_P denote the coordinates of the point of contact on the plane $O\xi\eta$. The kinematic relationships yield

$$\xi_P^{\boldsymbol{\cdot}} = (r\theta^{\boldsymbol{\cdot}}) \sin\psi - \mu\varphi^{\boldsymbol{\cdot}} \cos\psi,\quad \eta_P^{\boldsymbol{\cdot}} = -\mu\varphi^{\boldsymbol{\cdot}} \sin\psi - r\theta^{\boldsymbol{\cdot}} \cos\psi \tag{2.13}$$

Let $\zeta_P = \xi_P + i\eta_P$. Then, by (2.13),

$$\zeta_P^{\boldsymbol{\cdot}} = -(\mu\varphi^{\boldsymbol{\cdot}} + ir\theta^{\boldsymbol{\cdot}})\, e^{i\psi} \tag{2.14}$$

The function $-(\mu\varphi^{\boldsymbol{\cdot}} + ir\theta^{\boldsymbol{\cdot}})\, e^{i\psi_*(t)}$ is τ-periodic. Expanding it in Fourier series $\Sigma a_n e^{i\lambda_1 nt}$, we infer from (2.14) that

$$\zeta_P = \zeta_0 + \sum_{-\infty}^{\infty} \frac{a_n}{i(n\lambda_1 + \lambda_2)}\, e^{i(n\lambda_1+\lambda_2)t} \tag{2.15}$$

where ζ_0 is a constant.

If $n\lambda_1 + \lambda_2 \neq 0$ for integer n, then $\zeta_P = \zeta_0 + \chi(t)\, e^{i\lambda_2 t}$, where $\chi(t)$ is a τ-periodic function. Introducing a moving reference system rotating at angular velocity $-\lambda_2$ about the point ζ_0, we see that in the moving system the point $\zeta_P(t)$ will move periodically along a closed curve $\zeta_P = \chi(t)$. Thus, in the fixed plane $O\xi\eta$ the point of contact will move in a rather complicated way: it will move periodically along a certain curve, which in turn rotates like a rigid body about a fixed point, at constant angular velocity.

However, if the resonance relationships $n\lambda_1 + \lambda_2 = 0$ hold, then the mean motion of ξ_P and η_P may well fail to vanish (the question of the independence of the frequencies will be considered later), i.e., the body of revolution may go off to infinity.

We will now discuss the behaviour of the trajectories of motion in the phase space T^*M. If $\mathbf{c} \notin \Sigma$, then any connected component of the level set of the first integrals is diffeomorphic to a three-dimensional torus /15/. Indeed, this set is a compact oriented three-dimensional manifold. It admits of three pairwise commuting linearly independent tangent vector fields. The first is defined by the right-hand sides of Eqs.(1.1), and the coordinate curves corresponding to ψ and φ serve as integral curves for the other two fields. Hence it follows that this is a torus (or several tori). The trajectories of motion are straight lines winding uniformly around the tori.

We now return to the equations of motion and briefly consider the question of whether they are Hamiltonian. In /16/ we showed how to construct coordinates

$$\pi_1 = \theta,\ \pi_2 = \psi + \Psi(h, P_1, P_2, \theta),\ \pi_3 = \varphi + \Phi(h, P_1, P_2, \theta)$$

(subsequently replacing h, P_1, P_2 by appropriate expressions in terms of $\mathbf{p}, \mathbf{q}$) which, together with the momentum transformation (2.5), reduce Eqs.(1.1) to the form of ordinary Hamilton equations with Hamiltonian H. Here there is no invariant set of motions with constant angle θ on which $\Gamma_\psi = \Gamma_\varphi = 0, p_\theta = 0$ and the equations are now in Hamiltonian form with Hamiltonian $H = H(p_\psi, p_\varphi, \psi, \varphi)$. Unfortunately, these coordinates cannot be used to construct a canonical atlas, i.e., to introduce canonical coordinates globally. The reason for this situation is that the coordinates π_2 and π_3 are not necessarily angular, since

$$\oint \Psi d\theta \neq 0 \,(\mathrm{mod}\ 2\pi),\ \oint \Phi d\theta \neq 0\ (\mathrm{mod}\ 2\pi)$$

for almost all value of h, P_1, P_2 (the integration is performed along curves $H_0 = \mathrm{const}$).

Thus, we have the following.

Proposition 2. The phase space T^*M of the problem, or even any of its invariant subspaces (other than the manifold of motions with constant angle θ), cannot be given a symplectic structure in such a way that the equations of motion (1.1) assume a Hamiltonian form with Hamiltonian H.

Proof. (by reductio ad absurdum). If this were possible, the variables $\boldsymbol{\pi}, \mathbf{P}$ would be symplectic. However, one cannot construct a symplectic atlas with these coordinates.

However, if we consider the problem of reducing the equations of motion to Hamiltonian form with some other Hamiltonian H_*, there is a simple solution (the theorem on rectification of the phase flow). There exist coordinates $\mathbf{w} \bmod 2\pi, \mathbf{I}$ (see below) in terms of which the equations of motion have the form

$$\mathbf{w}^{\cdot} = \lambda(\mathbf{I}),\ \mathbf{I}^{\cdot} = 0 \tag{2.16}$$

If $|\partial\lambda/\partial\mathbf{I}| \neq 0$, these equations are Hamiltonian, the Hamiltonian being

$$H_* = \lambda_1^2(\mathbf{I}) + \lambda_2^2(\mathbf{I}) + \lambda_3^2(\mathbf{I})$$

Note that in the case of a dynamically and geometrically symmetric body Eqs.(1.1) do not admit of Chaplygin's reducing muliplier /1/. Indeed, if such a multiplier existed, it would be equal (apart from a constant factor) to the last Jacobi multiplier $\delta^{-1}(\theta)$ and in the case of a uniform sphere it would be a constant. However, it is well-known that even for a uniform sphere $\Gamma \neq 0$ in Eqs.(1.1).

3. In the neighbourhood of the invariant tori of the problem, we introduce the variables $\mathbf{w} \bmod 2\pi, \mathbf{I}$ (analogues of the action-angle variables in Hamiltonian mechanics), in terms of which the phase flow is rectified, i.e., the equations of motion have the form of (2.16).

Put $I_2 = P_1, I_3 = P_2$. The variables w_1, I_1 are defined as the usual action-angle variables for a one-dimensional Hamiltonian system (2.6):

$$I_1(H_0, I_2, I_3) = (2\pi)^{-1} \oint \operatorname{sign} p_\theta [2(H_0 - \Pi)\omega_1^{-1}]^{1/2} d\theta \tag{3.1}$$

$$w_1 = \operatorname{sign} p_\theta \lambda_1 \int [2(H_0 - \Pi)\omega_1]^{-1/2} d\theta, \quad \lambda_1 \equiv \partial H_0(I)/\partial I_1 \tag{3.2}$$

The paths of integration in (3.1) are the iso-energetic curves $H_0 = \text{const}$ in the θ, p_θ plane.

Differentiating both sides of (3.1) with respect to H_0 and using the expresssion (2.8) for the period of motion τ, we have

$$\partial I_1/\partial H_0 = \tau/(2\pi) > 0 \tag{3.3}$$

Since $\partial I_1/\partial H_0 > 0$, Eq.(3.1) is solvable for H_0 and the formula for λ_1 is well defined.

The angular coordinates w_2, w_3 are defined as follows /16/:

$$w_2 = \psi + W_2(H_0, I_2, I_3, \theta),\ w_3 = \varphi + W_3(H_0, I_2, I_3, \theta) \tag{3.4}$$

The functions W_2, W_3 are so chosen that $w_2^{\cdot} = \lambda_2, w_3^{\cdot} = \lambda_3$. Differentiating (3.4) with respect to time, we obtain

$$W_2 = \int \frac{\lambda_2 - (\psi^{\cdot})^*}{(\theta^{\cdot})^*} d\theta = \operatorname{sign} p_\theta \int \frac{\lambda_2 - \Delta^{-1}(\rho_1 I_2 + \rho_2 I_3)}{\sqrt{2(H_0 - \Pi)\omega_1}} d\theta \tag{3.5}$$

$$W_3 = \int \frac{\lambda_3 - (\varphi^{\cdot})^*}{(\theta^{\cdot})^*} d\theta = \operatorname{sign} p_\theta \int \frac{\lambda_3 - \Delta^{-1}(\rho_3 I_2 + \rho_4 I_3)}{\sqrt{2(H_0 - \Pi)\omega_1}} d\theta \tag{3.6}$$

The asterisk in (3.5) and (3.6) means that the quantity in parentheses should be replaced by its expression in terms of H_0, I_2, I_3.

The frequencies λ_2 and λ_3 are so chosen that the coordinates w_2, w_3 are angular, i.e.,

$$\oint W_2 d\theta = \oint W_3 d\theta = 0 \tag{3.7}$$

Hence $\lambda_2 = \Delta\psi/\tau, \lambda_3 = \Delta\varphi/\tau$, where $\Delta\psi$ is the angle through which the line of nodes QE rotates about the point Q in a time equal to the oscillation period of the angle θ; $\Delta\varphi$ is the angle through which the body rotates about its axis of symmetry in the same time τ. Thus the frequencies $\lambda_1, \lambda_2, \lambda_3$ here are the same as in Sect.2.

Clearly, the variables $\mathbf{w}$ (as well as $\boldsymbol{\pi}$) are not uniquely defined (for example, one can add an arbitrary function of the variables H_0, I_2, I_3).

The procedure used to introduce the variables $\mathbf{w}, \mathbf{I}$ is essentially taken from Hamiltonian mechanics. As the problem is non-holonomic, we have $\lambda_2 \neq \partial H_0/\partial I_2,\ \lambda_3 \neq \partial H_0/\partial I_3$.

4. We will now investigate the non-degeneracy of the frequencies λ in this problem. Put $J = |\partial\lambda/\partial\mathbf{I}|$. In the case of an arbitrary dynamically and geometrically symmetric body,

verification of the condition $J \not\equiv 0$ involves considerable technical difficulties, mainly because the matrix $X(\theta)$ is not known. We shall verify the condition below for the case of a body bounded by a spherical surface. Obviously, $J \not\equiv 0$ in the most general case as well, except when $f = r > 0$. Then the motion has a single frequency (when $A = C$) or two frequencies $(A \neq C)$.

Proposition 3. Let $f(\theta) = r + d\cos\theta, A \neq C, d \neq 0$. Then $J \not\equiv 0$.

Proof /16/.

$$\frac{\partial \lambda}{\partial \mathbf{I}} = \frac{\partial(\lambda_1, \lambda_2, \lambda_3)}{\partial(\tau, \Delta\psi, \Delta\varphi)} \mathbf{K}_1 \frac{\partial(u, I_3, v)}{\partial(H_0, I_2, I_3)} \frac{\partial(H_0, I_2, I_3)}{\partial(I_1, I_2, I_3)} = \frac{2\lambda_1^2}{(\tau I_3)^3} \mathbf{K}_1, \quad \mathbf{K}_1 = \frac{\partial(\tau, \Delta\psi, \Delta\varphi)}{\partial(u, I_3, v)}$$

Here $u = 2(H_0 - mgr) I_3^{-2}$, $v = I_2 I_3^{-1}$. We express $\tau, \Delta\psi, \Delta\varphi$ explicitly in terms of u, I_3, v

$$\tau = \oint \frac{d\theta}{(\theta^{\cdot})^*}, \quad \Delta\psi = \oint \frac{(\psi^{\cdot})^*}{(\theta^{\cdot})^*} d\theta, \quad \Delta\varphi = \oint \frac{(\varphi^{\cdot})^*}{(\theta^{\cdot})^*} d\theta$$

$$(\theta^{\cdot})^* = I_3 \{\omega_1 [u - 2I_3^{-2} mgd \cos\theta - A^{-1} \sin^{-2}\theta (d_1 v^2 + d_2 v + d_3)]^{1/2}$$

$$(\psi^{\cdot})^* = I_3 A^{-1} \sin^{-2}\theta (v + \varkappa_1 \alpha), \quad (\varphi^{\cdot})^* = I_3 A^{-1} [-(v + \alpha\varkappa_1) \sin^{-2}\theta \cos\theta + \alpha\alpha_1]$$

$$\varkappa_1 = \varkappa r^{-1}, \quad \alpha = [\delta\delta(0)]^{-1}, \quad \alpha_1 = AC^{-1}, \quad d_1 = 1 + m\varkappa^2 A^{-1},$$

$$d_2 = 2\varkappa_1 \alpha \delta^2, \quad d_3 = \delta^{-2}(0)(\varkappa_1^2 + \alpha_1 \sin^2\theta)$$

Investigation of the behaviour of $\tau, \Delta\psi, \Delta\varphi$ as $I_3 \to \infty$ $(v \to 0)$ /13/ shows that $J \not\equiv 0$, thus proving the proposition.

We now consider the motion of a body similar in shape and mass distribution to a dynamically and geometrically symmetric body. Then

$$\zeta = f(\theta) + \varepsilon f_1(\theta, \varphi), \; B = A(1 + \varepsilon) \; (0 \leqslant \varepsilon \ll 1)$$

In the unperturbed case $(\varepsilon = 0)$ we obtain a conditionally periodic motion on three-dimensional tori. By generalizations of Kolmogorov's theorem /18-20/ to reversible systems (system (1.1) is reversible), the tori on which the frequencies are "sufficiently strongly incommensurate" do not disappear, but are only slightly shifted in phase space, the motion remaining conditionally periodic. These invariant tori form a set of positive measure. Consequently, for the majority of initial conditions the phase portrait in the θ, p_θ plane will differ only slightly from the phase portrait of the unperturbed problem. In particular, for the majority of initial conditions the range of variations of the angle of nutation changes only slightly, and the same is true of the nature and position of the curves traced out by the point of contact on the plane $Q\xi_1\eta_1$ and the surface bounding the body.

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DIFFUSION SPREADING OF LOCALIZED HYDRODYNAMIC DISTURBANCES UNDER THE ACTION OF RANDOM FORCES*

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The effect of a time-dependent random force on fluid flow may be found by changing to a non-inertial coordinate system. It is shown that, under the action of a Gaussian random force, initially localized disturbances undergo spreading of a diffusion type. Explicit analytic solutions are given for the interior wave soliton under the action of a random force. It is shown that, in the presence of a soliton, the growth of velocity pulsing may either increase or moderate.

1. The evolution of a wide class of one-dimensional disturbances of the velocity field of the flow $u(x, t)$ in hydrodynamics is described by the general non-linear equation /1/

$$u_t + uu_x + \int\limits_{-\infty}^{\infty} dyF(y-x)\,u_{yy}(y,t) = f \tag{1.1}$$

When there are no external force $(f = 0)$ the Cauchy problem for the homogeneous equation can sometimes be solved by means of reduction to a linear problem, and as a result of the balance of non-linearity, dispersion, and dissipation, the existence of selfpreserving non-linear fields (solitons and shock waves) is possible. In particular, when $F(x) = -\mu\delta(x)$, we obtain Burgers' equation, which, under the Hopf-Cole replacement, reduces to the linear equation of diffusion. For

$$F(x) \sim -\delta'(x), \quad P\frac{1}{x}, \quad P\left(\operatorname{ctg}\frac{\pi x}{2h} - \operatorname{sgn} x\right)$$

the equations are respectively, completely integrable Korteweg- de Vries equations, Benjamin-Ono equations, and the equations of the interior waves in a basin of finite depth (the symbol P indicates that the singular integrals are to be taken in the sense of the principal value). The reducibility to a linear problem in these cases is also well-known /2/.

With regard to the non-uniform Eq.(1.1), by using the equivalence of the action of the spatially homogeneous force $f(t)$ and of a suitable acceleration of the coordinate system, the soltuion of (1.1) for $u(x, t)$ can be reduced by the change of variables

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